ON THE COHOMOLOGY OF THE REAL GRASSMANN COMPLEXES AND THE CHARACTERISTIC CLASSES OF n-PLANE BUNDLES(1)

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1. **Introduction.** This paper deals with two things: first the cohomology of the real Grassmann spaces; and secondly, relations between the various characteristic classes of n-plane bundles. Let ζ^n be a real n-plane bundle over a paracompact space B. With ζ^n we associate the Stiefel-Whitney characteristic class

$$W(\zeta^n) = 1 + W_1(\zeta^n) + \cdots + W_n(\zeta^n),$$

where $W_i(\zeta^n) \in H^i(B; Z_2)$ (Z=integers, Z_r=integers mod r). We refer to Milnor [4] for the properties of n-plane bundles and for the axioms which the Stiefel-Whitney classes satisfy.

Similarly, let ω^n be a complex n-plane bundle over a paracompact space X. With ω^n we associate the Chern characteristic class

$$c(\omega^n) = 1 + c_1(\omega^n) + \cdots + c_n(\omega^n),$$

where $c_i(\omega^n) \in H^{2i}(X; Z)$. The Chern characteristic classes are axiomatized by Hirzebruch in [3].

The remaining characteristic classes we consider are the Pontrjagin classes. Given a real n-plane bundle ζ, the Pontrjagin class $p(\zeta)$ is defined by

$$p_i(\zeta) = (-1)^i c_{2i}(\zeta_c) \qquad (i = 0, 1, \cdots, [n/2]), \tag{1.1}$$

where ζ_c is the complexification of the bundle ζ (see [4, XII] for details). Thus, $p_i(\zeta) \in H^{4i}(B; Z)$, where B is the base of ζ.

For the results which follow we will need the following relationship between the Pontrjagin and Chern classes. Suppose that ω^n is a complex n-plane bundle. Each fibre of ω^n is then an n-dimensional complex vector space. By restricting attention to the real numbers as scalars, we see that each fibre is also a $2n$-dimensional *real* vector space. Thus, ω^n gives rise to a real $2n$-plane bundle, ω^n_R. We then have the following result relating the classes of these two bundles (see Wu [9] and [4, §XII]):

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(1.2) THEOREM.

$$p_i(\omega_R^n) = (c_i(\omega^n))^2 + \sum_{j=0}^{i-1} (-1)^{i+j} 2c_j(\omega^n)c_{2i-j}(\omega^n).$$

Grassmann complexes. In order to obtain further results on the characteristic classes we will study the cohomology of the real Grassmann complexes, G_n $(n=1, 2, \cdots)$. These are defined as follows. Let R^∞ denote the countable direct sum of the real numbers: i.e., a point in R^∞ is a sequence $(r_1, r_2, \cdots, r_n, \cdots)$, $r_i \in R$, where all but a finite number of r_i are zero. Define G_n to be the set of all n-dimensional linear subspaces of R^∞. Milnor shows that G_n is in fact a CW-complex (see [4, V]), and exhibits the cell-decomposition.

With each complex G_n we associate a canonical n-plane bundle, γ^n, by defining

$$\gamma^n = (E_n, \pi_n, G_n), \tag{1.3}$$

where

$$E_n = \{(V, e) \mid V \text{ is an } n\text{-dimensional subspace of } R^\infty, e \in V\};\ \pi_n(V, e) = V.$$

Notice that $G_1 = P^\infty$, the infinite dimensional real projective space. The importance of the spaces G_n is underlined by the following classification theorem (see [5, 19.3]):

(1.4) THEOREM. *Let X be a paracompact space. Then, the equivalence classes of real n-plane bundles over X are in* 1-1 *correspondence with the homotopy classes of maps $X \to G_n$.*

The cohomology of the space G_n is therefore important, since it is mapped into the cohomology of the base space of each real n-plane bundle. We state here the facts which are known about this cohomology.

(1.5) THEOREM. *Let n be a positive integer and set*

$$W_i = W_i(\gamma^n) \qquad (i = 0, 1, \cdots, n).$$

Then,

$$H^*(G_n; Z_2) = Z_2[W_1, W_2, \cdots, W_n].$$

For a proof see Chern [2] and Milnor [4].

(1.6) THEOREM. *Let p be an odd prime and set*

$$\bar{p}_i = p_i(\gamma^n) \bmod p \qquad (i = 0, 1, \cdots, [n/2]).$$

Then,

$$H^*(G_n; Z_p) = Z_p[\bar{p}_1, \cdots, \bar{p}_q] \qquad (q = [n/2]).$$

This follows from the work of Borel (see [1]). As a consequence of Theorem (1.6) we have[2]

(1.7) THEOREM. *Let R_0 be the rational numbers and denote by ρ_0 the cohomology homomorphism induced by the inclusion $Z \rightarrow R_0$. Set*

$$\overset{0}{p}_i = \rho_0(p_i(\gamma^n)) \qquad (i = 0, 1, \cdots, [n/2]).$$

Then,

$$H^*(G_n; R_0) = R_0[\overset{0}{p}_1, \cdots, \overset{0}{p}_q] \qquad (q = [n/2]).$$

2. **Statement of results.** We discuss the cohomology of the complexes G_n from two points of view: first, we give results which partially characterize the algebraic structure of the cohomology groups; and secondly, we give mappings which isomorphically embed the cohomology of G_n into the cohomology of simpler spaces.

We begin by describing certain cohomology[3] homomorphisms. Define

$$(2.1) \qquad \theta_i: Z_2 \rightarrow Z_{2^i} \; \rho_j: Z \rightarrow Z_j \quad (i = 1, 2, \cdots; j = 2, 3, \cdots)$$

by

$$\theta_i(1 \bmod 2) = 2^{i-1} \bmod 2^i; \qquad \rho_j(1) = 1 \bmod j.$$

Let δ_* be the Bockstein coboundary associated with the exact sequence

$$0 \rightarrow Z \xrightarrow{2} Z \xrightarrow{\rho_2} Z_2 \rightarrow 0,$$

and set

$$(2.2) \qquad \beta_k = \rho_k \delta_* \qquad (k = 2, 3, \cdots).$$

Recall that the coboundary β_2 is a derivation—i.e., $\beta_2(uv) = \beta_2(u)v + u\beta_2(v)$ for any mod 2 cohomology classes u and v.

Now let n be a fixed positive integer and consider the cohomology ring $H^*(G_n; Z_2)$. Since β_2 is a derivation, the kernel of β_2 is a subring of $H^*(G_n; Z_2)$. In §3 we show that

(2.3) PROPOSITION. *Kernel $\beta_2 = P_2 \oplus T_2$ (vector space direct sum) where*

$$P_2 = Z_2[W_2^2, W_4^2, \cdots, W_{2q}^2] \quad \textit{and} \quad T_2 = \textit{Image}\, \beta_2.$$

Here $W_i = W_i(\gamma^n)$ $(i = 1, 2, \cdots, n)$ and $q = [n/2]$. Proposition 2.3 leads to a description of the integral cohomology ring of G_n.

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[2] For an independent proof, see [4, XV].

[3] If θ is any coefficient group homomorphism, we will denote by the same symbol the cohomology group homomorphism induced by θ.

THEOREM A. *$H^*(G_n; Z) = P \oplus T$ (group direct sum) where*
$P = Z[p_1, \cdots, p_q]$ ($p_i = p_i(\gamma^n)$), T = ideal of torsion elements.

We next describe the cohomology of G_n with coefficients mod 2^i, $i = 1, 2, \cdots$. We may regard $H^*(G_n; Z_2)$ simply as a vector space over the field Z_2. Since β_2 is a linear vector homomorphism, the kernel of β_2 is a linear subspace of $H^*(G_n; Z_2)$. We obtain a direct sum (vector) splitting of $H^*(G_n; Z_2)$ by choosing a complement to Kernel β_2. That is, we write

$$H^*(G_n; Z_2) = \text{Kernel } \beta_2 \oplus S_2 \text{ (vector direct sum).}$$

Let P_2 be the summand defined in Proposition (2.3), and let Q_2 be the summand of $H^*(G_n; Z_2)$ spanned by all the monomials (in $W_1, \cdots, W_n$) not belonging to P_2. Clearly, $\beta_2 Q_2 \subset Q_2$; and hence, Image $\beta_2 \subset Q_2$. Thus, we may choose the summand S_2 so that Image $\beta_2 \oplus S_2 \subset Q_2$. We assume that such a choice has been made.

For each positive integer i define

$$S_{2^i} = \theta_i(S_2), \qquad T_{2^i} = \rho_{2^i}(T),$$

where T is given in Theorem A. S_{2^i} and T_{2^i} are each subgroups of $H^*(G_n; Z_{2^i})$, and we in fact have

THEOREM B. *$H^*(G_n; Z_{2^i}) = P_{2^i} \oplus T_{2^i} \oplus S_{2^i}$ ($i = 1, 2, \cdots$) where*
(a) *$P_{2^i} = \rho_{2^i}(P) = Z_{2^i}[\bar{p}_1, \cdots, \bar{p}_q]$;*
(b) *ρ_{2^i} maps T isomorphically onto T_{2^i};*
(c) *θ_i maps S_2 isomorphically onto S_{2^i}.*

Here $\bar{p}_j = \rho_{2^i}(p_j(\gamma^n))$ ($j = 1, \cdots, q$), and the splitting is simply a module splitting over the ring Z_{2^i}. These results do not completely characterize the cohomology of G_n, since the algebraic structure of the rings T, T_{2^i} is not given, nor is any information about S_2, S_{2^i} given. In a subsequent paper Theorems A and B will be used to complete the characterization of $H^*(G_n; Z)$. For the purpose of this paper, however, this lack of information is compensated for in the following way.

Let P^∞ and $P^\infty(C)$ be respectively the real and complex infinite dimensional projective spaces. Let n be a fixed, positive integer, and set $q = [n/2]$. In §4 we define a real n-plane bundle ξ^n over $(P^\infty)^n$ (n-fold cartesian product) and a real n-plane bundle η^n over $(P^\infty(C))^q$ (q-fold cartesian product). Let

$$(2.4) \qquad f_n: (P^\infty)^n \to G_n, \qquad g_n: (P^\infty(C))^q \to G_n$$

be mappings which induce the respective bundles ξ^n, η^n from the universal bundle γ^n over G_n (see 1.4). We then have the following results: consider the cohomology homomorphisms f_n^*, g_n^* on the group $H^*(G_n; Z)$ (see Theorem A).

THEOREM A′. (a) *f_n^* restricted to T is a monomorphism*(4), *mapping T into $H^*((P^\infty)^n; Z)$;*

(b) *g_n^* restricted to P is a monomorphism, mapping P into $H^*((P^\infty(C))^q; Z)$.*

Next, consider the homomorphisms f_n^*, g_n^* on the group

$$H^*(G_n; Z_{2^i}), \text{ for } i = 1, 2, \cdots ; \text{(see Theorem B)}.$$

THEOREM B′. (a) *f_n^* restricted to $T_{2^i} \oplus S_{2^i}$ is a monomorphism, mapping $T_{2^i} \oplus S_{2^i}$ into $H^*((P^\infty)^n; Z_{2^i})$;*

(b) *g_n^* restricted to P_{2^i} is a monomorphism, mapping P_{2^i} into*

$$H^*((P^\infty(C))^q; Z_{2^i}).$$

The original motivation for this research was to understand the result of Wu (see [8] and Theorem C below). This led to the above results, and in §10 we make use of Theorem B′ to give a new proof of Wu's theorem. To state this, let $\mathfrak{P}_2$ be the Pontrjagin square cohomology operation (see [6]), and Sq^i the Steenrod square cohomology operation ($i=0, 1, \cdots$). Let θ_2 and β_4 be the operations defined respectively in (2.1) and (2.2). We then(5) have:

THEOREM C (WU). *Let ξ be a real n-plane bundle over a space X. Set $\overline{W}_i = W_i(\xi)$ ($i=1, 2, \cdots, n$), and $\bar{p}_j = p_j(\xi)$ ($j=1, 2, \cdots, [n/2]$). Then,*

(a) $$\mathfrak{P}_2(\overline{W}_{2i+1}) = \beta_4 Sq^{2i}\overline{W}_{2i+1} + \theta_2(\overline{W}_1 Sq^{2i}\overline{W}_{2i+1});$$

(b) $$\mathfrak{P}_2(\overline{W}_{2i}) = \rho_4\bar{p}_i + \theta_2\left[\overline{W}_1 Sq^{2i-1}\overline{W}_{2i} + \sum_{j=0}^{i-1} \overline{W}_{2j}\overline{W}_{4i-2j}\right].$$

The paper is organized as follows: in §3 we give the proof of Proposition 2.3, while in §§4, 5 we establish some facts about the cohomology of the spaces $(P^\infty)^n$ and $(P^\infty(C))^q$. In §6 we discuss the cohomology of a certain type of space of which the complexes G_n are a special case. Finally, the proof of Theorem A is given in §7, the proof of Theorem B in §8, the proofs of Theorems A′ and B′ in §9, and the proof of Theorem C in §10.

I should like to thank Professor S. S. Chern for putting me in touch with Wu's paper and for suggesting to me that a different proof of Wu's theorem might be found.

3. **A direct sum splitting of $H^*(G_n; Z_2)$.** The proof of Proposition 2.3 will follow at once from a theorem about polynomial algebras. Let n be a positive integer, and denote by $A^{(n)}$ the polynomial algebra over Z_2 in indeterminates $w_1, \cdots, w_n$ with unit 1. Regarding $A^{(n)}$ as a vector space over Z_2, we

(4) By monomorphism we mean a homomorphism whose kernel is zero.

(5) This statement of the theorem differs from that given by Wu, as he uses a slightly different definition of the Pontrjagin classes. For an interesting application of the theorem see R. Bott and J. Milnor, *On the parallelizability of the spheres*, Bull. Amer. Math. Soc. vol. 64 (1958) pp. 87–89.

assume in addition that $A^{(n)}$ has a linear endomorphism β with the following properties:

$$\beta(uv) = \beta(u)v + u\beta(v) \qquad u, v \in A^{(n)}; \tag{3.1}$$

$$\beta \circ \beta = 0; \tag{3.2}$$

$$\beta(w_1) = w_1^2, \qquad \beta(w_{2i}) = w_{2i+1} \quad (1 \leqq i \leqq [(n-1)/2]), \tag{3.3}$$
$$\beta(w_n) = w_1 w_n \quad (n \text{ even}).$$

By 3.1 we see that β is a derivation. Thus, the kernel of β is a subring of $A^{(n)}$; and Image β is an ideal in Kernel β. Setting $K^{(n)} = \text{Kernel } \beta$ we then have:

(3.4) THEOREM. *$K^{(n)} = P^{(n)} \oplus T^{(n)}$ (vector space direct sum) where $P^{(n)} = Z_2[w_2^2, w_4^2, \cdots, w_{2q}^2]$ and $T^{(n)} = \text{Image } \beta$ ($q = [n/2]$).*

We first show how Theorem 3.4 implies Proposition 2.3. Let γ^n be the n-plane bundle over G_n defined in 1.3, and let $W_1, \cdots, W_n$ be the Stiefel-Whitney classes of γ^n. Let β_2 be the Bockstein coboundary defined in 2.2. Then (see [8]),

$$\beta_2(W_{2i}) = W_1 W_{2i} + W_{2i+1}, \qquad \beta_2(W_{2i+1}) = W_1 W_{2i+1}.$$

We make a change of basis for $H^*(G_n; Z_2)$ as follows (see [1]). For $1 \leqq i \leqq [n/2]$, $1 \leqq j \leqq [(n-1)/2]$, set

$$W_1^* = W_1, \qquad W_{2i}^* = W_{2i}, \qquad W_{2j+1}^* = W_1 W_{2j} + W_{2j+1}. \tag{3.5}$$

Clearly, we continue to have:

$$H^*(G_n; Z_2) = Z_2[W_1^*, \cdots, W_n^*].$$

Furthermore,

$$\beta_2(W_1^*) = W_1^{*2}, \qquad \beta_2(W_{2i}^*) = W_{2i+1}^* \qquad (1 \leqq i \leqq [(n-1)/2]),$$
$$\beta_2(W_n^*) = W_1^* W_n^* \quad (n \text{ even}).$$

Thus, $H^*(G_n; Z_2)$ with generators $W_1^*, \cdots, W_n^*$ and linear homomorphism β_2 is a polynomial algebra of the type $A^{(n)}$ defined in 3.1–3.3. Since $W_{2i} = W_{2i}^*$, Proposition 2.3 follows at once from Theorem 3.4.

The proof of Theorem 3.4 falls into 2 cases: n even and n odd. Suppose first that n is an even integer, say $n = 2q$ ($q \geqq 1$). The proof is then a consequence of Theorem 2 in [7]. For setting

$$w_{2i} = x_i, \qquad w_{2i+1} = y_i, \qquad (1 \leqq i \leqq q-1),$$
$$w_1 = u, \qquad w_{2q} = v,$$

we obtain the algebra B_q with the derivation β^*, considered in §3 of [7].

We now prove Theorem 3.4 for the case $n=2q-1$ ($q\geqq 1$). To simplify the notation set $A^{(2q)}=A$. We regard $A^{(2q-1)}$ as the subspace, A', of A spanned by all the monomials in $w_1, w_2, \cdots, w_{2q-1}$. Set I = ideal of A generated by w_{2q}. Clearly, $A=A'\oplus I$ (vector direct sum). Since $\beta w_{2q}=w_1w_{2q}$, it is clear that $\beta(I)\subset I$, $\beta(A')\subset A'$. Let $\beta'=\beta$ restricted to A'. Then,

$$\text{Kernel } \beta' = \text{Kernel } \beta \cap A', \qquad \text{Image } \beta' = \text{Image } \beta \cap A'.$$

Since

$$P^{(2q)} \cap A' = Z_2[w_2^2, w_4^2, \cdots, w_{2q-2}^2] = P^{2(q-1)},$$

we have proved Theorem 3.4 for the case $n = 2q-1$ and hence for all n.

4. **The spaces** $(P^\infty)^n$, $(P^\infty(C))^q$. In order to continue the study of the cohomology of the complexes G_n, we first study the cohomology of two simpler spaces, stating here some well-known results (e.g., see [4]).

We have noted earlier that when $n=1$, then $G_1=P^\infty$, the infinite real projective space, and that the canonical 1-plane bundle γ^1 over P^∞ has Stiefel-Whitney class

$$W(\gamma^1) = 1 + a,$$

where a generates $H^1(P^\infty; Z_2)$.

Let n be a fixed positive integer. We will need the characteristic classes of a certain bundle ξ^n over $(P^\infty)^n$; that is, the n-fold cartesian product of the space P^∞. Let π_i: $(P^\infty)^n \rightarrow P^\infty$ be the ith projection map. Let γ_i^1 be the bundle over $(P^\infty)^n$ induced by π_i from γ^1, and let $a_i=\pi_i^*(a)$ ($i=1, 2, \cdots, n$). Define

$$\xi^n = \gamma_1^1 \oplus \cdots \oplus \gamma_n^1. \tag{4.1}$$

Then, (see [4, VI]),

(4.2) $W_i(\xi^n)=\sigma_i$, *the ith elementary symmetric function in the classes* $a_1, \cdots, a_n$.

In order to determine the Pontrjagin classes of ξ^n we first determine the Chern classes of ξ^n_c, the complexification of ξ^n. By reasoning very similar to that used to prove (4.2), one shows that

(4.3) $c_i(\xi^n_c)=\tau_i$, *the ith elementary symmetric function in the classes* $b_1, \cdots, b_n$, *where* $b_i=\delta_*(a_i)$.

Therefore, by Definition 1.1 we have:

$$p_i(\xi^n) = \tau_{2i}, \tag{4.4}$$

since $-\tau_{2i}=\tau_{2i}$.

We now look at $P^\infty(C)$, the infinite complex projective space. In a way entirely analogous to 1.3 one defines a canonical complex 1-plane bundle

ω^1 over $P^\infty(C)$. Also, $c(\omega^1)=1+d$, where d is the canonical generator of $H^2(P^\infty(C); Z)$. As is well known, $H^*(P^\infty(C); Z)$ is a polynomial ring in the class d.

Let n again be a fixed integer >1; set $n=2q+\epsilon$, where $\epsilon=0$ or 1. As above let π_j: $(P^\infty(C))^q \to P^\infty(C)$ be the projection map on the jth factor ($j=1, 2, \cdots, q$). Let $d_j=\pi_j^*(d)$. Then, $H^*((P^\infty(C))^q; Z)$ is a polynomial ring in the classes $d_1, \cdots, d_q$. Let ω_j^1 be the complex 1-plane bundle over $(P^\infty(C))^q$ induced from ω^1 by π_j (see 1.4 in the complex case). Then, exactly as in (4.2), one has

(4.5) $c_j(\theta^q)=\mu_j$, *the jth elementary symmetric function in the classes* $d_1, \cdots, d_q$, *where* $\theta^q=\omega_1^1\oplus \cdots \oplus\omega_q^1$.

Let θ_R^q be the real $2q$-plane bundle obtained from θ^q. Using Theorem (1.2) one obtains

(4.6) $p_i(\theta_R^q)=\nu_i$, *the ith elementary symmetric function in the classes* $d_1^2, \cdots, d_q^2$.

The purpose of introducing the complex $(P^\infty(C))^q$ is to study the cohomology of G_n. To this end we introduce a real n-plane bundle η^n over $(P^\infty(C))^q$ by defining

$$\eta^n = \begin{cases} \theta_R^q, & \text{if } n = 2q, \\ \theta_R^q \oplus \tau^1, & \text{if } n = 2q+1 \end{cases} \tag{4.7}$$

where τ^1 is the trivial real 1-plane bundle over $(P^\infty(C))^q$. We then have:

(4.8) Lemma. $p(\eta^n)=p(\theta_R^q)$, $W(\eta^n)=W(\theta_R^q)$.

The proof follows at once from the multiplicative properties of the Pontrjagin and Stiefel-Whitney classes.

(4.9) Corollary. $p_i(\eta^n)=\nu_i$ (*see* 4.6).

5. **Cohomology operations and characteristic classes.** From the standpoint of applications it is obviously very useful to have as many relations as possible between the various characteristic classes. It is the purpose of this section to exhibit such relations. These are known results, but we give simple proofs based on the methods of [4].

(5.1) Theorem. *Let ζ^n be a real n-plane bundle, and let ρ_2 be the cohomology homomorphism induced by the factor map $Z\to Z_2$. Then,* (*see* [9]),

$$\rho_2(p_i(\zeta^n)) = W_{2i}(\zeta^n)^2 \qquad (1 \leqq i \leqq [n/2]).$$

Proof. As is the case for any theorem concerning characteristic classes, it suffices to prove the theorem in the universal case $\zeta^n=\gamma^n$, the canonical n-plane bundle over G_n. Let p_i, W_j be the respective characteristic classes for the bundle γ^n.

In §4 we defined an n-plane bundle ξ^n over $(P^\infty)^n$ and calculated its characteristic classes in 4.2 and 4.4. Using the notation of 4.2, we stated a well known fact (e.g., see [4, VII]) which we will need later.

(5.2) LEMMA. *Let f_n be a map from $(P^\infty)^n$ to G_n which induces ξ^n. Then, f_n^* maps $H^*(G_n;\ Z_2)$ monomorphically into $H^*((P^\infty)^n;\ Z_2)$; the image of f_n^* is $Z_2[\sigma_1, \cdots, \sigma_n]$.*

Our goal is to show that $\rho_2(p_i) = W_{2i}^2$. We do this by making a calculation in the bundle ξ^n. We write

$$p_i(\xi^n) = \sum \delta_*(a_1) \cdot \cdots \cdot \delta_*(a_{2i}),$$
$$W_{2i}(\xi^n) = \sum a_1 \cdots a_{2i},$$

where the summation indicates that we are taking respective symmetric functions with the indicated elements as first terms. It follows that

$$\begin{aligned} \rho_2 p_1(\xi^n) &= \sum \rho_2\delta_*(a_1) \cdots \rho_2\delta_*(a_{2i}) \\ &= \sum \beta_2(a_1) \cdots \beta_2(a_{2i}) \\ &= \sum a_1^2 \cdots a_{2i}^2. \end{aligned}$$

Here we have used the fact that $\beta_2 = Sq^1$, and $Sq^1(u) = u^2$ when dimension $u = 1$. But (see 4.2),

$$(W_{2i}(\xi^n))^2 = (\sum a_1 \cdots a_{2i})^2 = \sum a_1^2 \cdots a_{2i}^2,$$

since $2a_j = 0$ ($j = 1, \cdots, n$). Thus,

$$\rho_2 p_i(\xi^n) = W_{2i}(\xi^n)^2 \qquad (1 \leqq i \leqq [n/2]).$$

Now,

$$f_n^* \rho_2(p_i) = \rho_2 p_i(\xi^n), \qquad f_n^*(W_{2i})^2 = W_{2i}(\xi^n)^2.$$

Therefore, $f_n^*[\rho_2(p_i) - W_{2i}^2] = 0$. But by 5.2, f_n^* is a monomorphism. Consequently,

$$\rho_2(p_i) = W_{2i}^2,$$

as was to be proved.

Now, let ω^n be an n-dimensional complex plane bundle; and denote by ω_R^n the real $2n$-dimensional plane bundle determined by ω^n. We then have the following result (see [9]):

(5.3) THEOREM. $\rho_2(c_i(\omega^n)) = W_{2i}(\omega_R^n)$ $(i = 1, \cdots, n)$.

Proof. We use the formula stated in Theorem 1.2: namely,

$$p_i(\omega_R^n) = (c_i(\omega^n))^2 + \sum_{j=0}^{i-1} (-1)^{i+j} 2c_j(\omega^n) c_{2i-j}(\omega^n).$$

Thus,

$$\rho_2(p_i(\omega_R^n)) = \rho_2(c_i(\omega^n))^2 = (\rho_2 c_i(\omega^n))^2.$$

Using 5.1, we have

$$(W_{2i}(\omega_R^n))^2 = (\rho_2 c_i(\omega^n))^2.$$

The fact that this implies

$$W_{2i}(\omega_R^n) = \rho_2 c_i(\omega^n),$$

follows at once using the universal bundle over $(P^\infty(C))^n$; the details are left to the reader.

The relations between characteristic classes obtained so far have used only the coefficient group homomorphism ρ_2. In order to get further relations on the classes W_i, we must use cohomology operations defined on cohomology classes mod 2. We will use two types of such operations: the Steenrod squares and the Pontrjagin square.

For future reference we state a special case of a result of Wu:

(5.4) THEOREM (WU). *Let Sq^i be the Steenrod square mapping $H^q(X; Z_2)$ to $H^{q+i}(X; Z_2)$. Let ζ be an n-plane bundle. Then,*

$$Sq^i W_{i+1}(\zeta) = \sum_{j=0}^{i} W_j(\zeta) W_{2i+1-j}(\zeta).$$

In §10 we will determine the Pontrjagin square on the classes W_j. Here we prove a special case which will be needed in that section:

(5.5) THEOREM. *Let ω^n be a complex n-plane bundle. Let $\mathfrak{P}_2$ be the Pontrjagin square mapping $H^{2q}(X; Z_2)$ to $H^{4q}(X; Z_4)$. Let ρ_4, θ_2 be the homomorphisms defined in §2. Then,*

$$\mathfrak{P}_2(W_{2i}(\omega_R^n)) = \rho_4(p_i(\omega_R^n)) + \theta_2\left[\sum_{j=0}^{i-1} W_{2j}(\omega_R^n) W_{4i-2j}(\omega_R^n)\right].$$

Proof. We need the following fact about the Pontjragin square (see [6, Theorem I and 2.1]).

(5.6) *Let $u \in H^{2q}(X; Z)$. Then, $\mathfrak{P}_2\rho_2(u) = \rho_4(u^2)$.*

Now, let $c = c(\omega^n)$, $p = p(\omega_R^n)$, $W = W(\omega_R^n)$. By 5.3, $\rho_2(c_i) = W_{2i}$. Thus,

$$\mathfrak{P}_2(W_{2i}) = \mathfrak{P}_2(\rho_2(c_i)) = \rho_4(c_i^2). \tag{5.7}$$

Recalling Theorem 1.2, we know

$$\rho_4(c_i^2) = \rho_4(p_i) - \rho_4\left(\sum_{j=0}^{i-1} (-1)^{i+j} 2c_jc_{2i-j}\right).$$

Let η_2 be induced by the factor map $Z_4 \to Z_2$. Then,

$$2\rho_4 = \theta_2\eta_2\rho_4 = \theta_2\rho_2.$$

Thus,

$$\pm 2\rho_4(c_jc_{2i-j}) = \theta_2\rho_2(c_jc_{2i-j}) = \theta_2(\rho_2(c_j)\rho_2(c_{2i-j})) = \theta_2(W_{2j}W_{4i-2j}).$$

Consequently,

$$(5.8) \qquad \rho_4(c_i^2) = \rho_4(p_i) + \theta_2\left(\sum_{j=0}^{i-1} W_{2j}W_{4i-2j}\right).$$

Combining 5.7 and 5.8 we have our result.

6. **The cohomology of certain spaces.** In order to continue the study of the cohomology of the complexes G_n, we digress to study the cohomology of a general class of spaces which includes the spaces G_n. That is, we study spaces X with the property that if T denotes the torsion subgroup of $H^*(X; Z)$, then there is a prime number p such that $pT=0$. We see by [1, Theorem 24.7], that the complexes G_n are examples of such a space, with $p=2$.

We need a bit of notation which we will keep throughout this section. Let X be a fixed space of the type described above. Set

$$A = H^*(X; Z), \qquad A_n = H^*(X; Z_n), \qquad (n = 2, 3, \cdots).$$

Let ρ_n be the homomorphism of A to A_n induced by the factor map $Z \to Z_n$. Let β_0 be the coboundary associated with the exact sequence

$$(6.1) \qquad 0 \to Z \xrightarrow{p} Z \xrightarrow{\rho_p} Z_p \to 0.$$

Set $\beta_n = \rho_n \circ \beta_0$, and notice that

$$(6.2) \qquad \beta_p \circ \beta_p = 0, \qquad \beta_p \circ \rho_p = 0.$$

Using the exactness of 6.1 together with 6.2, the following facts are easily proved.

(6.3) LEMMA. *Let $T \subset A$ be the torsion subgroup of A. Then ρ_p restricted to T is a monomorphism mapping T into A_p.*

(6.4) LEMMA.

$$\beta_0(A_p) = T.$$

(6.5) LEMMA.

$$\rho_p(T) = \text{Image } \beta_p \subset A_p; \qquad \rho_p(A) = \text{Kernel } \beta_p \subset A_p.$$

Now, A_p may be regarded as a vector space over Z_p. Since β_p is a homomorphism of A_p into itself, Kernel β_p is a linear subspace of A_p; set V_p $=$Kernel β_p. By 6.5, $V_p=\rho_p(A)$. Since A_p is a vector space, V_p is a direct summand; let S_p be a complement to V_p. That is,

$$A_p = V_p \oplus S_p, \tag{6.6}$$

as a vector space.

Let i be a fixed integer $\geqq 1$. Our goal is to determine the structure of $A_{p^i}=H^*(X;\, Z_{p^i})$. Let θ_i be the homomorphism from A_p to A_{p^i} induced by the homomorphism from Z_p to Z_{p^i} which sends 1 mod p into p^{i-1} mod p^i. Set

$$V_{p^i} = \rho_{p^i}(A), \qquad S_{p^i} = \theta_i(S_p).$$

Then,

(6.7) Theorem. $A_{p^i}=V_{p^i}\oplus S_{p^i}$ *(group direct sum)*.

We precede the proof by several lemmas. To begin with, consider the commutative diagram

$$\begin{array}{llcl} (*) & 0 \to Z \xrightarrow{p} & Z & \xrightarrow{\rho_p} Z_p \to 0 \\ & \quad\;\; \| & \downarrow p^{i-1} & \quad \downarrow \theta_i \\ (**) & 0 \to Z \xrightarrow{p^i} & Z & \xrightarrow{\rho_{p^i}} Z_{p^i} \to 0. \end{array}$$

Let β_0, δ_0 be the Bockstein coboundary operators associated respectively with the exact sequences (*) and (**). By the commutativity of the diagram one has

$$\beta_0 = \delta_0\theta_i. \tag{6.8}$$

We use this to show:

(6.9) Lemma.

$$V_{p^i} \cap S_{p^i} = 0.$$

Proof. Let $u\in V_{p^i}\cap S_{p^i}$; i.e., $u=\rho_{p^i}(x)$ for $x\in A$, and $u=\theta_i(y)$, for $y\in S_p$. Therefore,

$$\delta_0(u) = \delta_0\rho_{p^i}(x) = 0,$$

by the exactness of (**). But we also have,

$$\delta_0(u) = \delta_0\theta_i(y) = \beta_0(y), \text{ by 6.8.}$$

Thus $\beta_0(y)=0$. Therefore, $\beta_p(y)=\rho_p\beta_0(y)=0$. That is, $y\in$Kernel $\beta_p=V_p$. However, by hypothesis, $y\in S_p$, and by 6.6, $V_p\cap S_p=0$. Therefore, $y=0$ and hence $u=\theta_i(y)=0$; this completes the proof.

One obtains similarly,

(6.10) LEMMA. *δ_0 maps S_{p^i} isomorphically onto T.*

As an immediate consequence we have

(6.11) LEMMA. *θ_i is an isomorphism mapping S_p onto S_{p^i}.*

We now give the proof of Theorem 6.7. Notice that $\delta_0(A_{p^i})\subset T$. Thus, by Lemma 6.10, we may define an inverse to δ_0 restricted to S_{p^i}. Let ϵ_0: $T\to S_{p^i}$ be the inverse; i.e., $\delta_0\epsilon_0=$identity. For any $u\in A_{p^i}$, set $u_2=\epsilon_0\delta_0(u)$. Then,

$$\delta_0(u - u_2) = \delta_0(u) - \delta_0\epsilon_0\delta_0(u) = \delta_0(u) - \delta_0(u) = 0.$$

Thus, by exactness of (**), $u-u_2=\rho_{p^i}(v)$ for some $v\in A$. That is, $u-u_2\in V_{p^i}$. Set $u_1=u-u_2$. Then,

$$u = u_1 + u_2, \qquad u_1 \in V_{p^i},\ u_2 \in S_{p^i}.$$

Since $S_{p^i}\cap V_{p^i}=0$, this splitting is unique. Thus, $A_{p^i}=V_{p^i}\oplus S_{p^i}$ as was asserted.

7. **The proof of Theorem** A. Denote by P the subring of $H^*(G_n; Z)$ generated by $p_1, \cdots, p_q$. We first show that P is in fact a polynomial ring. Let R_0 be the rational numbers, and let ρ_0 be the cohomology homomorphism induced by the inclusion $Z\subset R_0$. Let ϕ be a polynomial in $Z[x_1, \cdots, x_q]$ and suppose that $\phi(p_1, \cdots, p_q)=0$. We show that $\phi\equiv 0$. Let ϕ_0 be the image of ϕ in $R_0[x_1, \cdots, x_q]$. Then, $0=\rho_0(\phi(p_1, \cdots, p_q))=\phi_0(\overset{0}{p}_1, \cdots, \overset{0}{p}_q)$. But by Theorem 1.7, this implies that $\phi_0\equiv 0$. Now the injection $Z[x_1, \cdots, x_q]\to R_0[x_1, \cdots, x_q]$ is a monomorphism. Thus, $\phi\equiv 0$; and hence, P is a polynomial ring.

Since every element of P has infinite order, it is clear that $P\cap T=0$. Thus, Theorem A is proved when we show

(7.1) LEMMA. *Let $u\in H^*(G_n; Z)$. Then, there are elements $u_1\in P$, $u_2\in T$ such that $u=u_1+u_2$.*

To prove this consider the exact sequence

$$0 \to T \xrightarrow{\iota} H^*(G_n; Z) \xrightarrow{\rho_0} H^*(G_n; R_0), \tag{7.2}$$

where ι is the inclusion homomorphism. It follows from Theorem (1.7) that for any class $u\in H^*(G_n; Z)$ we have

$$\rho_0(u) = \phi(\overset{0}{p}_1, \cdots, \overset{0}{p}_q),$$

where $\phi\in R_0[x_1, \cdots, x_q]$. If $\phi\equiv 0$, then $u\in T$ by the exactness of sequence (7.2). Suppose then that $\phi\not\equiv 0$. Let $a_1, \cdots, a_N$ be the nonzero coefficients of ϕ, where each a_i is a rational number written in lowest form. Set

$$d_i = \text{denominator of the fraction } a_i,$$

and let

$$m = \text{least common multiple of } d_1, \cdots, d_N.$$

Then, $ma_1, ma_2, \cdots, ma_N$ are all integers. Furthermore, if p is a prime number dividing m, then:

(7.3) *There is at least one integer ma_i which p does not divide.*

Let $\psi = m\phi$; then ψ has integer coefficients and may be regarded as an element of $Z[x_1, \cdots, x_q]$. Let

$$u_1 = \psi(p_1, \cdots, p_q).$$

Clearly,

$$\rho_0(u_1) = \psi(\overset{0}{p}_1, \cdots, \overset{0}{p}_q) = m\phi(\overset{0}{p}_1, \cdots, \overset{0}{p}_q) = m\rho_0(u).$$

Therefore, $\rho_0(mu - u_1) = 0$. Hence, by the exactness of the sequence (7.2), there is a class $u_2 \in T$ such that

$$mu = u_1 + u_2.$$

The proof of Lemma (7.1) consists simply in showing that $m = 1$. Suppose, to the contrary, that m contains a prime factor p. Let ρ_p be the cohomology homomorphism induced by the reduction $Z \to Z_p$. Then, $\rho_p(mu) = 0$; i.e.,

$$\rho_p(u_1) + \rho_p(u_2) = 0. \tag{7.4}$$

We must consider two cases:

Case I. $p > 2$. Then, $\rho_p(u_2) = 0$, since $2T = 0$. Hence, (7.4) implies that $\rho_p(u_1) = 0$. Recall that $u_1 = \psi(p_1, \cdots, p_q)$. Now by (7.3), the polynomial ψ has at least one coefficient which is not divisible by p. Thus, $\psi_p \neq 0$, where ψ_p denotes the image of ψ in $Z_p[x_1, \cdots, x_q]$. But,

$$\rho_p(u_1) = \rho_p(\psi(p_1, \cdots, p_q)) = \psi_p(\bar{p}_1, \cdots, \bar{p}_q),$$

where $\bar{p}_i = \rho_p(p_i)$. From Theorem (1.6) we see that $\psi_p(\bar{p}_1, \cdots, \bar{p}_q) \neq 0$. Thus, $\rho_p(u_1) \neq 0$, which is a contradiction. Consequently, m is not divisible by any odd prime.

Case II. $p = 2$. Then, (7.4) implies that

$$\rho_2(u_1) = \rho_2(u_2).$$

Now from Theorem (5.1) and Proposition (2.3) it is clear that $\rho_2(P) = P_2$. Also, by Lemma (6.5), $\rho_2(T) = T_2$. Hence, $\rho_2(u_1) \in P_2 \cap T_2$. But by Proposition (2.3), $P_2 \cap T_2 = 0$. Thus, $\rho_2(u_1) = 0$. But now the same argument as that given in Case I may be used to obtain a contradiction—using Theorem (1.5) in this case instead of Theorem (1.6), (together with the fact that $\rho_2(p_i) = W_{2i}^2$). Hence, assuming that the integer m contains a prime factor always leads to a contradiction. Therefore, $m = 1$, which proves Lemma (7.1) and hence Theorem A.

8. **The proof of Theorem** B. Let us apply the results of §6 to the group $H^*(G_n; Z_{2^i})$ $(i=1, 2, \cdots)$. As in §2 we may choose a summand S_2 of $H^*(G_n; Z_2)$ such that

$$H^*(G_n; Z_2) = \text{Kernel } \beta_2 \oplus S_2 \text{ (vector direct sum).}$$

Thus, by Theorem 6.7, we have

$$H^*(G_n; Z_{2^i}) = V_{2^i} \oplus S_{2^i},$$

where $V_{2^i} = \rho_{2^i}(H^*(G_n; Z))$, $S_{2^i} = \theta_i(S_2)$, and the direct sum is a module splitting over the ring Z_{2^i}. Now, from Theorem A (and Lemma 6.5) it is clear that $V_{2^i} = \rho_{2^i}(P) \oplus \rho_{2^i}(T) = \rho_{2^i}(P) \oplus T_{2^i}$. From Lemma 6.11 we have that θ_i maps S_2 isomorphically onto S_{2^i}; and from Lemma 6.3 it follows that ρ_{2^i} maps T isomorphically onto T_{2^i}. Thus, Theorem B is proved when we show that $\rho_{2^i}(P)$ is a polynomial ring in $\rho_{2^i}(p_1), \cdots, \rho_{2^i}(p_q)$ $(q=[n/2])$. But that follows at once from Theorem A and the fact that $2T=0$.

Before going on to the proofs of Theorems A′ and B′, we remark one more property of the groups $H^*(G_n; Z_{2^i})$ which will be needed in §10. When $i=1$ we have the splitting

$$H^*(G_n; Z_2) = P_2 \oplus T_2 \oplus S_2,$$

given by Theorem B. Let θ_i be the homomorphism defined in 2.1. From Lemma 6.11, we know that θ_i maps S_2 isomorphically onto S_{2^i} $(i=1, 2, \cdots)$.

(8.1) Lemma. $\theta_i(T_2)=0$, $\theta_i(P_2) \subset P_{2^i}$ $(i=2, 3, \cdots)$.

Proof. We use the following fact: let η_{2^i} be the factor homomorphism $Z_{2^i} \to Z_2$. Then,

$$\theta_i \eta_{2^i}(u) = 2^{i-1}(u), \qquad u \in H^*(X; Z_{2^i}).$$

Thus,

$$\theta_i(T_2) = \theta_i \rho_2(T) = \theta_i \eta_{2^i} \rho_{2^i}(T) = 2^{i-1} T_{2^i} = 0,$$

since $i \geqq 2$ and $2T_{2^i}=0$.

Similarly, $\theta_i(P_2) = 2^{i-1} P_{2^{i-1}}$; which completes the proof of the lemma.

9. **The proofs of Theorems** A′ **and** B′.

Proof of Theorem A′. (a) We are to show that f_n^* restricted to T is a monomorphism; i.e., given $u \in T$ such that $f_n^*(u)=0$, we are to show that $u=0$. Let ρ_2 be the factor map defined in 2.1. Then, since $f_n^*(u)=0$, $\rho_2 f_n^*(n)=0$. But ρ_2 is natural; thus, $f_n^* \rho_2(u)=0$. Now, by 5.2, f_n^* is a monomorphism on $H^*(G_n; Z_2)$. Therefore, $\rho_2(u)=0$. But by Lemma 6.3, this implies that $u=0$.

(b) This follows at once from Corollary (4.9) and the fact that $P=Z[p_1, \cdots, p_q]$.

Proof of Theorem B′. (a) We give the proof in 2 parts. First, if $u \in T_{2^i}$ and $f_n^*(u)=0$, then the fact that $u=0$ follows by an argument entirely similar to (a) above. Suppose, secondly, that $u \in S_{2^i}$ and $f_n^*(u)=0$. We are to show that $u=0$. Since $u \in S_{2^i}$, by definition $u=\theta_i(v)$ for some class $v \in S_2$. Thus, $f_n^*(u)=0$ implies $f_n^*\theta_i(v)=0$; i.e., by naturality, $\theta_i f_n^*(v)=0$. Now consider the following commutative diagram,

$$
\begin{array}{ll}
(*) & 0 \longrightarrow Z \xrightarrow{2^{i-1}} Z \xrightarrow{\rho_{2^{i-1}}} Z_{2^{i-1}} \longrightarrow 0 \\
& \qquad\ \ \downarrow \rho_2 \qquad \downarrow \rho_{2^i} \qquad\qquad \| \\
(**) & 0 \longrightarrow Z_2 \xrightarrow{\theta_i} Z_{2^i} \xrightarrow{\xi} Z_{2^{i-1}} \longrightarrow 0,
\end{array}
$$

where ξ is the natural factor map. Let δ_0 be the coboundary associated with upper exact sequence, and δ_2 the coboundary associated with the lower exact sequence. By commutativity we have $\delta_2=\rho_2\delta_0$. Since $\theta_i f_n^*(v)=0$, by the exactness of (**) we have $f_n^*(v)=\delta_2(w)$, for some class $w \in H^*((P^\infty)^n;\ Z_{2^{i-1}})$; hence, $f_n^*(v)=\rho_2\delta_0(w)$. Therefore,

$$\beta_2 f_n^*(v) = \beta_2\rho_2\delta_0(w) = 0,$$

since $\beta_2\rho_2=0$. Thus, by naturality, $f_n^*\beta_2(v)=0$. But, by 5.2, f_n^* maps A_2 isomorphically into $H^*((P^\infty)^n;\ Z_2)$. Thus, $\beta_2(v)=0$; i.e., $v \in \text{Kernel } \beta_2$. But, by hypothesis, $v \in S_2$. From §2 we know Kernel $\beta_2 \cap S_2=0$; hence $v=0$. Therefore, $u=\theta_i(v)=0$, as was to be proved.

The proof of Theorem B′(b) is entirely similar. We leave the details to the reader.

10. **The proof of Theorem** C. We begin by giving some of the properties of the Pontrjagin square cohomology operation. This is a mapping $\mathfrak{P}_2$ from $H^q(X;\ Z_2)$ to $H^{2q}(X;\ Z_4)$, with the following properties:

(10.1) $\mathfrak{P}_2 f^*=f^*\mathfrak{P}_2$, where f^* is induced by a map f.

(10.2) $\eta_2\mathfrak{P}_2(u)=u^2$, where $u \in H^q(X;\ Z_2)$ and η_2 is induced by the factor homomorphism $Z_4 \to Z_2$.

(10.3) $\mathfrak{P}_2(u+v)=\mathfrak{P}_2(u)+\mathfrak{P}_2(v)+\theta_2(uv)$, where $u,\ v \in H^{2q}(X;\ Z_2)$ and θ_2 is defined in 2.1.

(10.4) $\mathfrak{P}_2(u)=\beta_4 Sq^{2q}(u)+\theta_2 Sq^{2q}\beta_2(u)$, where $u \in H^{2q+1}(X;\ Z_2)$, and $\beta_n=\rho_n\delta_*$ (see 2.2).

(10.5) $\mathfrak{P}_2(uv)=\mathfrak{P}_2(u)\mathfrak{P}_2(v)+\theta_2[Sq^{2r}(u)(v\beta_2(v))+(u\beta_2(u))Sq^{2s}(v)]$, where $u \in H^{2r+1}(X;\ Z_2)$, $v \in H^{2s+1}(X;\ Z_2)$.

The above properties are proved as follows: 10.1, 10.3 in [6, Theorem I]; 10.2 in [6, Theorem II]; 10.4 in [6, Proposition 7.7] and 10.5 in [6, p. 75]. With the exception of 10.4, they are also noted by Wu in [8].

As usual, it suffices to prove the theorem in the universal case, where $\xi^n=\gamma^n$, the n-plane bundle over G_n. We prove part (a) of Theorem C by first observing some facts:

(10.6) $$\beta_2(W_{2i+1}) = W_1 W_{2i+1}.$$

(10.7) $$Sq^r(uv) = Sq^r(u)v + Sq^{r-1}(u)\beta_2(v), \text{ if dimension } v = 1.$$

(10.8) $$\text{If } u, v \in H^*(X; Z_2), \quad \text{then} \quad \theta_2(u \cup \beta_2(v)) = \theta_2(\beta_2(u) \cup v).$$

(10.9) $$\beta_2 Sq^{2r+1} = 0 \qquad (r = 0, 1, \cdots).$$

Now by 10.4,

$$\mathfrak{P}_2(W_{2i+1}) = \beta_4 Sq^{2i}(W_{2i+1}) + \theta_2 Sq^{2i}\beta_2(W_{2i+1}).$$

Using 10.6-10.9 we have

$$\begin{aligned}\theta_2 Sq^{2i}\beta_2(W_{2i+1}) &= \theta_2 Sq^{2i}(W_1 W_{2i+1})\\ &= \theta_2[Sq^{2i}(W_{2i+1})W_1 + Sq^{2i-1}(W_{2i+1})\beta_2(W_1)]\\ &= \theta_2[W_1 Sq^{2i}(W_{2i+1})] + \theta_2[\beta_2 Sq^{2i-1}(W_{2i+1})W_1]\\ &= \theta_2[W_1 Sq^{2i}(W_{2i+1})].\end{aligned}$$

Thus,

$$\mathfrak{P}_2(W_{2i+1}) = \beta_4 Sq^{2i}(W_{2i+1}) + \theta_2(W_1 Sq^{2i} W_{2i+1}),$$

as was to be proved.

To prove part (b) of Theorem C we proceed as follows: by 10.2 we know that

$$\eta_2\mathfrak{P}_2(W_{2i}) = (W_{2i})^2.$$

By 5.1, we have

$$\rho_2(p_i) = (W_{2i})^2.$$

But, $\rho_2 = \eta_2\rho_4$, where η_2 is the factor map in the exact sequence

(*) $$0 \to Z_2 \xrightarrow{\theta_2} Z_4 \xrightarrow{\eta_2} Z_2 \to 0.$$

Thus,

$$\eta_2[\mathfrak{P}_2(W_{2i}) - \rho_4(p_i)] = 0.$$

Consequently, by the exactness of (*) there is a class $X \in H^*(G_n; Z_2)$ such that

(10.10) $$\mathfrak{P}_2(W_{2i}) = \rho_4(p_i) + \theta_2(X).$$

From Theorem B we have

$$X = X_1 + X_2 + X_3 \qquad (X_1 \in P_2,\ X_2 \in T_2,\ X_3 \in S_2);$$

and by 8.1, $\theta_2(X_2) = 0$; thus,

(10.11) $$\theta_2(X) = \theta_2(X_1) + \theta_2(X_3).$$

We prove Theorem C by determining the classes $\theta_2(X_1)$ and $\theta_2(X_3)$.

(10.12) LEMMA.

$$\theta_2(X_1) = 0,$$
$$\theta_2(X_3) = \theta_2\left[W_1Sq^{2i-1}W_{2i} + \sum_{j=0}^{i-1} W_{2j}W_{4i-2j}\right].$$

Proof. We first obtain the class $\theta_2(X_3)$. Let ξ^n be the bundle over $(P^\infty)^n$ defined in 4.1. Let $f: (P^\infty)^n \to G_n$ be the map which induces ξ^n from γ^n. Then,

$$f^*(W_i) = \sigma_i, \qquad f^*(p_i) = \tau_{2i}, \tag{10.13}$$

(see 4.2 and 4.4). We compute the operation $\mathfrak{P}_2$ in $H^*((P^\infty)^n; Z_2)$.

(10.14) LEMMA.

$$\mathfrak{P}_2(\sigma_{2i}) = \rho_4(\tau_{2i}) + \theta_2\left[\sigma_1Sq^{2i-1}\sigma_{2i} + \sum_{j=0}^{i-1} \sigma_{2j}\sigma_{4i-2j}\right].$$

The proof is given at the end of this section. We use Lemma 10.14 to compute the class $\theta_2(X_3)$. By 10.10 we have:

$$f^*\mathfrak{P}_2(W_{2i}) = f^*\rho_4(p_i) + f^*\theta_2(X).$$

Using 10.11, 10.13, and the fact that $\mathfrak{P}_2$, ρ_4 and θ_2 are cohomology operations, it follows that

$$\mathfrak{P}_2(\sigma_{2i}) = \rho_4(\tau_{2i}) + \theta_2(f^*X_1) + \theta_2(f^*X_3). \tag{10.15}$$

Comparing 10.14 and 10.15 we see that

$$\theta_2(f^*X_1) + \theta_2(f^*X_3) = \theta_2\left[\sigma_1Sq^{2i-1}\sigma_{2i} + \sum_{j=0}^{i-1} \sigma_{2j}\sigma_{4i-2j}\right]. \tag{10.16}$$

We first show

$$\theta_2(f^*X_1) = 0. \tag{10.17}$$

Since $X_1 \in P_2$ this is a consequence of the fact that

$$\theta_2(\hat{P}_2) = 0,$$

where $\hat{P}_2 = f^*(P_2)$. To show this recall that $P_2 = \rho_2(P)$. Therefore,

$$\theta_2(\hat{P}_2) = \theta_2(f^*P_2) = \theta_2(f^*\rho_2(P)) = \theta_2\rho_2(f^*(P)).$$

But,

$$\theta_2\rho_2 = \theta_2\eta_2\rho_4 = 2\rho_4.$$

Thus,

$$\theta_2\rho_2(f^*(P)) = 2\rho_4(f^*P) = \rho_4(2f^*(P).$$

Since $f^*(P)\subset H^*((P^\infty)^n; Z)$ and $2H^*((P^\infty)^n; Z)=0$, we have

$$\theta_2(\hat{P}_2) = \theta_2\rho_2(f^*(P)) = \rho_4(2f^*(P)) = 0,$$

which proves 10.17. From 10.13 we have that $f^*(W_i)=\sigma_i$. Using this, together, with 10.16, 10.17, we have

$$f^*\left[\theta_2(X_3) - \theta_2\left(W_1Sq^{2i-1}W_{2i} + \sum_{j=0}^{i-1} W_{2j}W_{4i-2j}\right)\right] = 0. \tag{10.18}$$

Consider the direct sum splitting given in Theorem B. By hypothesis, $X_3\in S_2$. But clearly, $(W_1Sq^{2i-1}W_{2i}+\sum_{j=0}^{i-1} W_{2j}W_{4i-2j})\in T_2\oplus S_2$. This follows from Theorem (5.4), the choice of the summand S_2, and the fact that $P_2=Z_2[W_2^2, \cdots, W_{2q}^2]$. Thus,

$$\theta_2\left[X_3 - \left(W_1Sq^{2i-1}W_{2i} + \sum_{j=0}^{i-1} W_{2j}W_{4i-2j}\right)\right] \in S_4,$$

since $\theta_2(T_2)=0$, $\theta_2(S_2)=S_4$, by 8.1 and 6.11. Finally, using 10.18 we have,

$$\theta_2\left[(X_3) - \left(W_1Sq^{2i-1}W_{2i} + \sum_{j=0}^{i-1} W_{2j}W_{4i-2j}\right)\right] = 0,$$

since f^* is a monomorphism on S_4 by Theorem B′ (a). Thus, the class $\theta_2(X_3)$ is determined in Lemma 10.12.

We have left to show that $\theta_2(X_1)=0$ to complete the proof of Lemma 10.12 and hence of Theorem C. To do this we use the model complex $(P^\infty(C))^q$, where $q=[n/2]$. Let $g: (P^\infty(C))^q\to G_n$ be the map which induces the bundle η^n over $(P^\infty(C))^q$ (see 4.7). Let

$$W_i(\eta^n) = \tilde{W}_i, \qquad p_i(\eta^n) = \tilde{p}_i \qquad (i = 0, 1, \cdots).$$

Then, by 4.8, we have

$$\tilde{W}_i = W_i(\theta_R^q), \qquad \tilde{p}_i = p_i(\theta_R^q),$$

where θ^q is the complex q-plane bundle over $(P^\infty(C))^q$ defined in 4.5.

Using Theorem 5.5 we have

$$\mathfrak{P}_2(\tilde{W}_{2i}) = \rho_4(\tilde{p}_i) + \theta_2\left(\sum_{j=0}^{i-1} \tilde{W}_{2j}\tilde{W}_{4i-2j}\right). \tag{10.19}$$

But,

$$\tilde{W}_i = g^*W_i, \qquad \tilde{p}_i = g^*p_i.$$

Therefore, applying g^* to 10.10, we obtain:

$$\mathfrak{P}_2(\tilde{W}_{2i}) = \rho_4(\tilde{p}_i) + \theta_2(g^*X_1) + \theta_2(g^*X_3). \tag{10.20}$$

Comparing 10.19 and 10.20, we see that:

$$(10.21) \qquad \theta_2(g^*X_1) + \theta_2(g^*X_3) = \theta_2\left(\sum_{j=0}^{i-1} \tilde{W}_{2j}\tilde{W}_{4i-2j}\right).$$

Now, by 10.12,

$$\theta_2(g^*X_3) = g^*\theta_2(X_3) = \theta_2\left(\tilde{W}_1 Sq^{2i-1}\tilde{W}_{2i} + \sum_{j=0}^{i-1} \tilde{W}_{2j}\tilde{W}_{4i-2j}\right).$$

But, $\tilde{W}_1 = 0$. Therefore,

$$\theta_2(g^*X_1) = g^*\theta_2(X_1) = 0, \text{ by 10.21.}$$

Now, by hypothesis, $X_1 \in P_2$; hence, $\theta_2(X_1) \in P_4$, by 8.2. But, by Theorem B′(b), g^* restricted to P_4 is a monomorphism. Thus, $g^*(\theta_2 X_1) = 0$ implies that $\theta_2(X_1) = 0$, as was asserted in 10.12. This, then, completes the proof of Lemma 10.12 and hence of Theorem C.

Proof of Lemma 10.14. The proof proceeds by induction on the integer n, the dimension of the bundle γ^n. When $n=1$, $(P^\infty)^n = P^\infty$. But $H^*(P^\infty; Z_2) = Z_2[a_1]$. Thus, $\sigma_{2i} = \tau_{2i} = 0$ $(i \geqq 1)$. Hence, both sides of 10.14 are zero and the lemma is true for $n=1$. Now, let n be a fixed integer >1. Suppose that we have proved Lemma 10.14 for all integers $j<n$. We prove the lemma for the integer n; and hence, by induction, for all integers. Let q: $(P^\infty)^n \to (P^\infty)^{n-1}$ be the map defined by $q(x_1, \cdots, x_{n-1}, x_n) = (x_1, \cdots, x_{n-1})(x_i \in P^\infty)$. Then, q^* maps $H^*((P^\infty)^{n-1}; Z_2)$ isomorphically into $H^*((P^\infty)^n; Z_2)$. We agree to identify $H^*((P^\infty)^{n-1}; Z_2)$ with its image under q^*; i.e., we set

$$H^*((P^\infty)^{n-1}; Z_2) = Z_2[a_1, \cdots, a_{n-1}], \text{ (see 4.1).}$$

Let ξ^{*n-1} be the bundle over $(P^\infty)^n$ induced by q from ξ^{n-1}. Clearly

$$W_i(\overset{*n-1}{\xi}) = \overset{*}{q}\,W_i(\overset{n-1}{\xi}) = \overset{*}{\sigma}_i,$$

$$p_i(\overset{*n-1}{\xi}) = \overset{*}{q}\,p_i(\overset{n-1}{\xi}) = \overset{*}{\tau}_{2i},$$

where σ_i^* is the ith elementary symmetric function in $a_1, \cdots, a_{n-1}$ and τ_j^* is the jth elementary symmetric function in $b_1, \cdots, b_{n-1}$ $(b_i = \delta_*(a_i))$. Using the monomorphism q^* we identify $W_i(\xi^{n-1})$ with σ_i^* and $p_i(\xi^{n-1})$ with τ_{2i}^*. Thus, by 10.14 and the inductive hypothesis we have

$$(10.22) \qquad \mathfrak{P}_2(\overset{*}{\sigma}_{2i}) = \rho_4(\overset{*}{\tau}_{2i}) + \theta_2\left[\overset{*}{\sigma}_1 Sq^{2i-1}\overset{*}{\sigma}_{2i} + \sum_{j=0}^{i-1} \overset{*}{\sigma}_{2j}\overset{*}{\sigma}_{4i-2j}\right].$$

Now, let σ_i, (resp. τ_j), be the symmetric functions in $a_1, \cdots, a_n$ (resp. $b_1, \cdots, b_n$) as before. Then, it is easily verified that

$$(10.23) \qquad \sigma_i = \overset{*}{\sigma}_{i-1}a_n + \overset{*}{\sigma}_i, \qquad \tau_j = \overset{*}{\tau}_{j-1}b_n + \overset{*}{\tau}_j$$

for $i, j=0, 1, \cdots$. Here we adopt the convention that

$$\sigma_{-1} = \tau_{-1} = \sigma_{n+1} = \tau_{n+1} = 0.$$

Combining 10.23 and 10.3 we obtain,

$$\mathfrak{P}_2(\sigma_{2i}) = \mathfrak{P}_2(\overset{*}{\sigma}_{2i-1}a_n) + \mathfrak{P}_2(\overset{*}{\sigma}_{2i}) + \theta_2(\overset{*}{\sigma}_{2i-1}\overset{*}{\sigma}_{2i}a_n).$$

By 10.5,

$$\mathfrak{P}_2(\overset{*}{\sigma}_{2i-1}a_n) = \mathfrak{P}_2(\overset{*}{\sigma}_{2i-1})\mathfrak{P}_2(a_n) + \theta_2\left[Sq^{2i-2}\overset{*}{\sigma}_{2i-1}(a_n\beta_2(a_n)) + \overset{*}{\sigma}_{2i-1}(\beta_2\overset{*}{\sigma}_{2i-1})Sq^0(a_n)\right].$$

Thus,

$$(10.24)\quad \mathfrak{P}_2(\sigma_{2i}) = \mathfrak{P}_2(\overset{*}{\sigma}_{2i-1})\mathfrak{P}_2(a_n) + \mathfrak{P}_2(\overset{*}{\sigma}_{2i}) + \theta_2\left[Sq^{2i-2}\overset{*}{\sigma}_{2i-1}(a_n\beta_2(a_n)) + \overset{*}{\sigma}_{2i-1}(\beta_2\overset{*}{\sigma}_{2i-1})Sq^0(a_n) + (\overset{*}{\sigma}_{2i-1}\overset{*}{\sigma}_{2i})a_n\right].$$

We prove Lemma 10.14 by analysing the right hand side of 10.24 term by term. Since $\sigma^*_{2i-1} = W_{2i-1}(\xi^{*n-1})$, by Theorem C (a) we have

$$(10.25)\quad \mathfrak{P}_2(\overset{*}{\sigma}_{2i-1}) = \beta_4 Sq^{2i-2}(\overset{*}{\sigma}_{2i-1}) + \theta_2(\overset{*}{\sigma}_1 Sq^{2i-2}\overset{*}{\sigma}_{2i-1}).$$

By methods similar to those in §5, one may easily show

$$(10.26)\quad \begin{aligned} &(a)\quad \delta_* Sq^{2i-2}(\overset{*}{\sigma}_{2i-1}) = (\overset{*}{\tau}_{2i-1}),\\ &(b)\quad \beta_4 Sq^{2i-2}(\overset{*}{\sigma}_{2i-1}) = \rho_4(\overset{*}{\tau}_{2i-1}).\end{aligned}$$

Next, consider the term $\mathfrak{P}_2(a_n)$ in the expression 10.24.

$$(10.27)\quad \mathfrak{P}_2(a_n) = \rho_4(b_n).$$

Proof. Since a_n is 1-dimensional (and hence odd), 10.4 gives

$$\mathfrak{P}_2(a_n) = \beta_4 Sq^0(a_n) + \theta_2 Sq^0(\beta_2 a_n) = \beta_4(a_n) + \theta_2\beta_2(a_n)$$

since $Sq^0 =$ identity. But $\beta_4 = \rho_4\delta_*$ and $\delta_*(a_n) = b_n$. Also, $\theta_2\beta_2 = 0$ by exactness. Thus,

$$\mathfrak{P}_2(a_n) = \rho_4\delta_*(a_n) + \theta_2\beta_2(a_n) = \rho_4(b_n),$$

as was asserted.

Finally, let $u \in H^*(X; Z_2)$ and $v \in H^*(X; Z)$ for any space X. Then, as is easily shown,

$$(10.28)\quad \theta_2(u)\rho_4(v) = \theta_2(u\rho_2(v)).$$

Combining 10.25 through 10.28 we have

$$(10.29)\qquad \mathfrak{P}_2(\overset{*}{\sigma}_{2i-1})\mathfrak{P}_2(a_n) = \rho_4(\overset{*}{\tau}_{2i-1}b_n) + \theta_2[(\overset{*}{\sigma}_1 Sq^{2i-2}\overset{*}{\sigma}_{2i-1})\beta_2(a_n)],$$

since $\rho_2(b_n) = \rho_2\delta_*(a_n) = \beta_2(a_n)$. Thus, combining 10.24, 10.29, and using the induction hypothesis 10.22 we have:

$$(10.30)\qquad \begin{aligned}\mathfrak{P}_2(\sigma_{2i}) = \rho_4(\tau_{2i}) + \theta_2\Big[&(\overset{*}{\sigma}_1 Sq^{2i-2}\overset{*}{\sigma}_{2i-1})\beta_2(a_n) + \overset{*}{\sigma}_1 Sq^{2i-1}\overset{*}{\sigma}_{2i} + \sum_{j=0}^{i-1}\overset{*}{\sigma}_{2j}\overset{*}{\sigma}_{4i-2j}\\ &+ (Sq^{2i-2}\overset{*}{\sigma}_{2i-1})(a_n)^3 + \overset{*}{\sigma}_1(\overset{*}{\sigma}_{2i-1})^2 a_n + \overset{*}{\sigma}_{2i-1}\overset{*}{\sigma}_{2i}a_n\Big].\end{aligned}$$

Here we have used the following facts:

(a) $\quad a_n(\beta_2(a_n)) = a_n(\overset{2}{a_n}) = \overset{3}{a_n};$

(b) $\quad \overset{*}{\sigma}_{2i-1}(\beta_2\overset{*}{\sigma}_{2i-1}) = \overset{*}{\sigma}_{2i-1}(\overset{*}{\sigma}_1\overset{*}{\sigma}_{2i-1}) = \overset{*}{\sigma}_1(\overset{*}{\sigma}_{2i-1})^2;$

(c) $\quad \overset{*}{\tau}_{2i-1}b_n + \overset{*}{\tau}_{2i} = \tau_{2i}.$

Comparing 10.30 and 10.14 we see that Lemma 10.14 is proved when we show

$$(10.31)\qquad \begin{aligned}&\text{(a)}\quad \theta_2[\overset{*}{\sigma}_1 Sq^{2i-2}\overset{*}{\sigma}_{2i-1})\beta_2(a_n) + (Sq^{2i-2}\overset{*}{\sigma}_{2i-1})(\overset{3}{a_n})\\ &\qquad\qquad + \overset{*}{\sigma}_1(\overset{*}{\sigma}_{2i-1})^2 a_n + \overset{*}{\sigma}_1(Sq^{2i-1}\overset{*}{\sigma}_{2i})\\ &\qquad\qquad = \theta_2[\sigma_1 Sq^{2i-1}\sigma_{2i} - (Sq^{2i-1}\overset{*}{\sigma}_{2i})a_n];\\ &\text{(b)}\quad \theta_2\left[\sum_{j=0}^{i-1}\overset{*}{\sigma}_{2j}\overset{*}{\sigma}_{4i-2j} + \overset{*}{\sigma}_{2i-1}\overset{*}{\sigma}_{2i}a_n\right]\\ &\qquad\qquad = \theta_2\left[\sum_{j=0}^{i-1}\sigma_{2j}\sigma_{4i-2j} - (Sq^{2i-1}\overset{*}{\sigma}_{2i})a_n\right].\end{aligned}$$

The proof of this is a simple mechanical verification using 10.23, 10.7 and Theorem 5.5. We leave the details to the reader.

11. **Appendix. The oriented case.** So far in this paper we have considered only the unoriented Grassmann complexes. We remark here an analogous result for the oriented Grassmann complexes, $\tilde{G}_n$ $(n=2, 3, \cdots)$. Let $\tilde{\gamma}^n$ be the classifying bundle over $\tilde{G}_n$ which is the analogue of γ^n, and let q be a positive integer.

(12.1) THEOREM. (a) *$H^*(\tilde{G}_{2q+1}; Z) = P\oplus T$, where $P = Z[p_1, \cdots, p_q]$ and T = ideal of torsion elements.* (b) *$H^*(\tilde{G}_{2q}; Z) = P^*\oplus T^*$, where $P^* = Z[p_1, \cdots, p_{q-1}, X]$ and T^* = ideal of torsion elements.*

Here $p_i = p_i(\tilde{\gamma}^n)$ and X is the Euler class of the bundle $\tilde{\gamma}^{2q}$ (see [4, VIII]). The proof of the theorem is very similar to the proof of Theorem A given in

§7. We use the analogues of Theorems (1.5)–(1.7) for the oriented case (see [1] and [4]), together with Corollary 1 of [7]. The details are left to the reader.

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